Rigid Body Rotations

A solution submission for problem three of the 2020 CAP problem competition

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1 Introduction

Problem three of the 2020 CAP problem competition asks for an explanation for the intermediate axis theorem (also referred to as the tennis racket theorem or the Dzhanibekov effect). The names given to this effect are historical, but it is purely a feature of Newton's second law when applied to rotating rigid bodies. Euler's equations for rigid body rotation govern the dynamics of these systems. His equations are first-order ordinary differential equations (ODEs), but they are by no means trivial to solve since they are coupled and non-linear. Rigid bodies and Euler's equations have been studied in detail for at least the last 200 years and exact solutions are known, although they involve elliptic functions. An easier way to visualize the problem was coined by Poinsot, known as Poinsot's construction. Instead of solving Euler's equations, Poinsot enforced conservation of energy and angular momentum as constraints on the solutions, and visualized the solution paths as the intersection between surfaces. I claim however that to understand the intermediate axis theorem, one does not need to directly solve Euler's equations, or even enforce conservation constraints. With some manipulation of Euler's equations, all of the dynamical features of the system can be visualized graphically and the instability of the intermediate axis can be proven with tools from stability theory.

2 Stability Theory

To prove the intermediate axis theorem, I will use some tools from stability theory and introduce them here. All of the information in this section and more can be found in [1]. Suppose a two dimensional system of ODEs can be written in the form

$$\frac{d\vec{x}}{dt} = A\vec{x}\,, \tag{1}$$

where A is a 2×2 matrix and $\vec{x}(t)$ is the solution to the system. The solution $\vec{x} = 0$ represents a "fixed point" in the system, where there is no change over time t in the solution (i.e. $d\vec{x}/dt = 0$). This fixed point is called stable if all trajectories limit to this point as $t \to \infty$. Stability theory allows for the classification of the stability of this point based on the eigenvalues of the matrix A.

Theorem *If A has eigenvalues λ_1 and λ_2, then $\vec{x} = 0$ is a locally asymptotically stable fixed point if* $\mathrm{Re}(\lambda_1) < 0$ *and* $\mathrm{Re}(\lambda_2) < 0$*, but is unstable if* $\mathrm{Re}(\lambda_1) > 0$ *and/or* $\mathrm{Re}(\lambda_1) > 0$.

The matrix A can be characterized in terms of its determinant $\Delta = \det(A)$ and its trace $\tau = \mathrm{Tr}(A)$. Given the following two dimensional matrix properties,

$$\Delta = \lambda_1 \lambda_2 , \quad \tau = \lambda_1 + \lambda_2 , \tag{2}$$

this theorem requires $\Delta > 0$ and $\tau < 0$ for a stable fixed point, with general cases summarized in Figure (1).

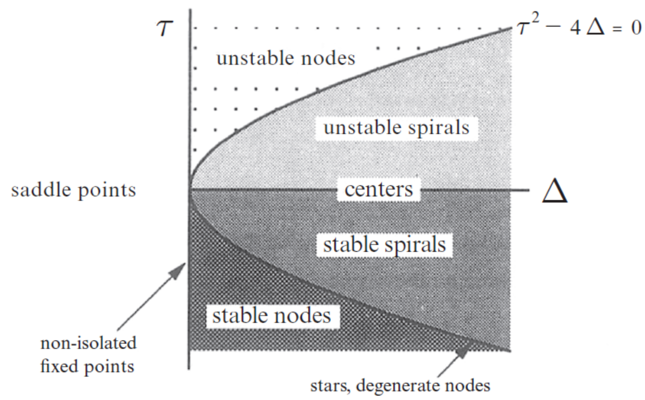

Figure 1: Stability chart of fixed points in linear two dimensional ODEs. Systems characterized in the lower right quadrant are called stable, while others are either unstable, periodic (centers), or non-isolated. [1]

This result will be useful as the local behavior of fixed points in nonlinear two dimensional systems can be approximated as linear using a Taylor expansion of the nonlinear system. This technique is called linearization. If instead there is a system that is characterized as

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) , \\ \frac{dy}{dt} &= g(x, y) , \end{aligned} \tag{3}$$

and it has a fixed point at (x_0, y_0), the system can be centered on this fixed point by substituting $u = x - x_0$, $v = y - y_0$, finding that

$$\begin{aligned} \frac{du}{dt} &= f(x_0 + u, y_0 + v) , \\ \frac{dv}{dt} &= g(x_0 + u, y_0 + v) . \end{aligned} \tag{4}$$

The Taylor expansion of this system then gives

$$\begin{aligned} \frac{du}{dt} &= f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} u + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} v + \mathcal{O}(u^2, v^2, uv) , \\ \frac{dv}{dt} &= g(x_0, y_0) + \left.\frac{\partial g}{\partial x}\right|_{(x_0,y_0)} u + \left.\frac{\partial g}{\partial y}\right|_{(x_0,y_0)} v + \mathcal{O}(u^2, v^2, uv) . \end{aligned} \tag{5}$$

Since (x_0, y_0) is a fixed point, by definition $f(x_0, y_0) = g(x_0, y_0) = 0$. Also, since I am only interested in the *local* stability of this fixed point (i.e. the region where $u \ll 1$ and $v \ll 1$), all higher order terms, $\mathcal{O}(u^2, v^2, uv)$, can be neglected. The linearized system has now been obtained,

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} \approx \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(x_0,y_0)} \begin{pmatrix} u \\ v \end{pmatrix} . \tag{6}$$

The above matrix of partial derivatives is called the Jacobian, and it can be treated like the matrix A earlier in this section for the purpose of classifying the local stability of fixed points in a general two dimensional system. For a stable point, the Jacobian must have $\Delta > 0$ and $\tau < 0$.

3 Euler's Equations for Rigid Body Rotation

To analyze the rotational dynamics of rigid bodies, I start with Euler's equations in three dimensions (in the absence of applied torque),

$$I\frac{d\vec{\omega}}{dt} + \vec{\omega} \times (I\vec{\omega}) = 0\,, \tag{7}$$

where I is the 3×3 inertia tensor and $\vec{\omega}$ is the angular velocity vector of the rigid body. Equation (7) is derived from Newton's second law, but adapted into the body-fixed reference frame. The reason for this is that it ensures that the inertia tensor of the body is not changing with time. First I will substitute the body-fixed angular momentum $\vec{L} = I\vec{\omega}$ to obtain

$$\frac{d\vec{L}}{dt} = \vec{L} \times \left(I^{-1}\vec{L}\right). \tag{8}$$

I will then choose a Cartesian basis (x, y, z) that is aligned with the angular momentum components such that the inertia tensor can be diagonalized as

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}. \tag{9}$$

At this point, I am motivated to try to reduce this system to a two dimensional one, because I have all of the knowledge of such systems from Section (2). To do this, I will represent this system of equations in terms of spherical coordinate variables (r, ϕ, θ). In terms of these variables, the angular momentum vector can be expressed as

$$\vec{L} = L[\cos(\phi)\sin(\theta)\hat{x} + \sin(\phi)\sin(\theta)\hat{y} + \cos(\theta)\hat{z}]\,. \tag{10}$$

I can then be substitute this into Equation (8) along with Equation (9) to obtain

$$\begin{aligned}\frac{1}{L^2}\frac{d\vec{L}}{dt} =& \frac{(I_y - I_z)}{I_y I_z}\sin(\theta)\cos(\theta)\sin(\phi)\hat{x} \\ &+ \frac{(I_z - I_x)}{I_x I_z}\sin(\theta)\cos(\theta)\cos(\phi)\hat{y} + \frac{(I_x - I_y)}{I_x I_y}\sin^2(\theta)\sin(\phi)\cos(\phi)\hat{z}\,.\end{aligned} \tag{11}$$

This Cartesian vector can then be expressed in terms of the spherical coordinate unit vectors $(\hat{r}, \hat{\phi}, \hat{\theta})$ by transforming it with the Cartesian-to-spherical transformation matrix,

$$T_{x\to r} = \begin{pmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \end{pmatrix}, \tag{12}$$

which results in

$$\frac{1}{L^2}\frac{d\vec{L}}{dt} = \frac{[I_x I_z + I_y I_z - 2I_x I_y + I_z(I_y - I_x)\cos(2\phi)]\sin(2\theta)}{4I_x I_y I_z}\hat{\phi} + \frac{(I_y - I_x)\sin(\phi)\cos(\phi)\sin(\theta)}{I_x I_y}\hat{\theta}\,. \tag{13}$$

Notice that the $d\vec{L}/dt$ vector has no component along the r-direction, as should be expected given that the magnitude of $\vec{L}$ should not change due to the conservation of angular momentum. Just by converting Euler's equations to spherical coordinates, we have just proven that his equations obey this crucial conservation law. The system has thus been successfully reduced to a two dimensional problem, and stability theory can be applied to understand the dynamics.

4 The Intermediate Axis Theorem Demonstrated

The result obtained in Equation (13) can now be plotted in a phase diagram, a plot that shows the direction of the vector field $d\vec{L}/dt$ as a function of (ϕ, θ). Keeping in mind that vector derivatives are scaled differently in spherical coordinates compared to Cartesian ones,

$$\frac{d\vec{L}}{dt} = \left(\frac{dL}{dt}\right)\hat{r} + \left(\frac{d\phi}{dt}\right)L\sin(\theta)\hat{\phi} + \left(\frac{d\theta}{dt}\right)L\hat{\theta}\,, \tag{14}$$

I will pick out just $d\phi/dt$ and $d\theta/dt$ for the plot. To ensure there is an intermediate axis, it is assumed that $I_x < I_y < I_z$. This defines the y-axis as the intermediate axis.

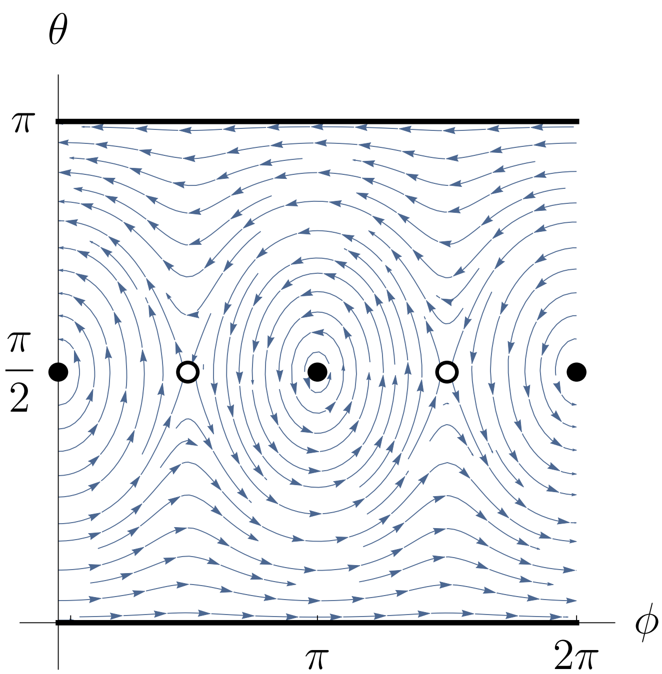

Figure 2: Phase diagram for rotating rigid bodies. Black dots and lines represent fixed points called centers. When the system starts near these points, the vector $\vec{L}$ will "orbit" around these points indefinitely, which is known as precession in rotational dynamics. The open circles represent unstable fixed points, where solution curves are repelled from being near for too long. I used Wolfram Mathematica to make this plot, and took $I_x = 1$, $I_y = 1.5$, $I_z = 2$.

Graphically, using Figure (2), it should be fairly obvious that the open circle points, the ones corresponding to the $\pm y$-axis directions, are unstable. This can be proven however, by computing the Jacobian matrix for this system, as demonstrated in Section (2),

$$J = L^2 \begin{pmatrix} \frac{2(I_x - I_y)}{I_x I_y}\cos(\theta)\cos(\phi)\sin(\theta)\sin(\phi) & \frac{[I_z(I_y - I_x)\cos(2\phi) - 2I_xI_y + I_xI_z + I_yI_z]}{2I_xI_yI_z}\cos(2\theta) \\ \frac{(I_y - I_x)}{I_xI_y}\sin(\theta)\cos(2\phi) & \frac{(I_y - I_x)}{I_xI_y}\cos(\theta)\sin(\phi)\cos(\phi) \end{pmatrix}, \tag{15}$$

and then evaluating it at the $+y$-axis, $(\phi_0, \theta_0) = (\pi/2, \pi/2)$. Evaluating instead at the $-y$-axis, $(\phi_0, \theta_0) = (3\pi/2, \pi/2)$, will give identical results. The result is

$$J = L^2 \begin{pmatrix} 0 & \frac{1}{I_z} - \frac{1}{I_y} \\ \frac{1}{I_y} - \frac{1}{I_x} & 0 \end{pmatrix}, \tag{16}$$

which has determinant and trace

$$\Delta = -L^4 \frac{(I_y - I_x)(I_z - I_y)}{I_x I_y^2 I_z}, \quad \tau = 0\,. \tag{17}$$

Both sets of parentheses in the numerator of Δ here are always positive, due to the $I_x < I_y < I_z$ constraint, forcing $\Delta < 0$. According to Figure (1), these fixed points are unstable saddle points. Evaluating instead at the $+x$-axis, $(\phi_0, \theta_0) = (0, \pi/2)$, the result is

$$\Delta = L^4 \frac{(I_y - I_x)(I_z - I_x)}{I_x^2 I_y I_z}, \quad \tau = 0\,. \tag{18}$$

Now $\Delta > 0$ and Figure (1) identifies this fixed point as a center. Trajectories here exhibit periodic motion and do not approach or depart from the fixed point as $t \to \infty$. There is identical behavior for the $-x$-axis and the $\pm z$-axes. To better understand the question at hand though, Equation (13) can be converted to be in terms of the angular velocity. The vector field $d\vec{L}/dt$ needs only to be rescaled by the inverse of the inertia tensor (after it is converted back to Cartesian coordinates),

$$\frac{d\vec{\omega}}{dt} = I^{-1} \frac{d\vec{L}}{dt}\,. \tag{19}$$

This allows for the projection of the vector field onto the Poinsot ellipsoid, the surface that the angular velocity vector $\vec{\omega}$ is constrained to move on, as in Figure (3).

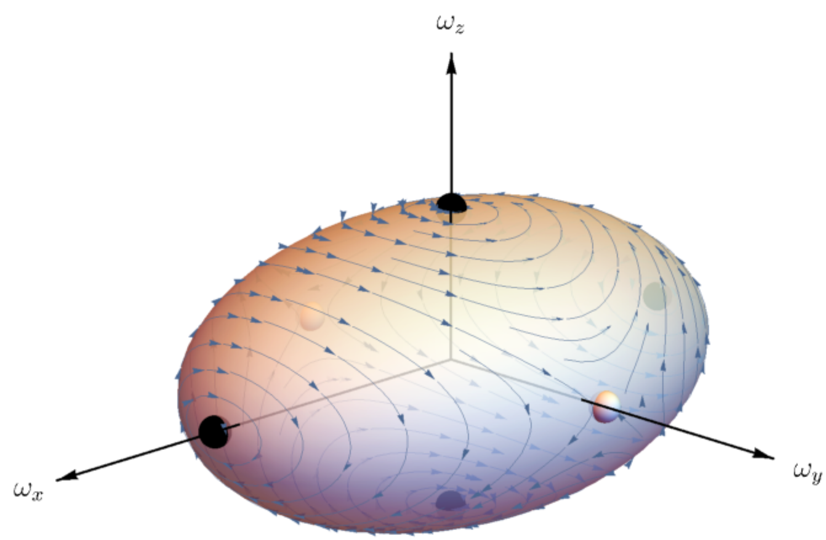

Figure 3: Phase diagram for rigid body rotation in three dimensional angular velocity space. Black dots represent the center fixed points, while white dots represent the unstable fixed points on the intermediate axis. Generated with Wolfram Mathematica, with $I_x = 1$, $I_y = 1.5$, $I_z = 2$.

So what happens when you try to flip a tennis racket? If you calculate the inertia tensor for the racket, you will find that trying to flip the head of the racket over the handle is equivalent to rotating a rigid body around the unstable intermediate axis. You could, in theory, rotate it on this axis, but it would quickly decay under any small perturbation, like trying to balance a ball on a horse's saddle. The racket will then settle onto a path that passes close to the unstable axis, but ultimately precesses around one of the stable axes. You can visually follow the path of precession using Figure (3), and such trajectories are called polhodes in the literature. Since the path does indeed pass close to the unstable axis anyway, it may appear to rotate around it briefly, but it will continue its precession around to the opposite axis, resulting in what looks like a "flip".

[1] Strogatz, S. (2015). "Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering." New York: CRC Press, Taylor and Francis Group.